DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU CTB2400)
Thursday August 13th 2015, 18:30-21:30

1. In this exercise we use the trapezoidal rule for the integration of the following initial value problem $y' = f(t, y)$ with $y(t_0) = y_0$:

$$w_{n+1} = w_n + \frac{\Delta t}{2}\left(f(t_n, w_n) + f(t_{n+1}, w_{n+1})\right) \qquad (1)$$

   (a) Show that the amplification factor of the trapezoidal rule is given by

$$Q(\Delta t\lambda) = \frac{1 + \frac{\Delta t\lambda}{2}}{1 - \frac{\Delta t\lambda}{2}}.$$

(1 pt.)

   (b) Give the order (+ proof) of the local truncation error of the trapezoidal rule for the test equation (hint the following series can be used: $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$ and $\frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k$). (3 pt.)

   (c) Do one step with the trapezoidal rule for the following initial value problem

$$y' = -2y + e^t, \text{ with } y(0) = 2,$$

   and step size $\Delta t = 1$. (2 pt.)

   (d) We consider the initial value problem:

$$y'' = -y' - \frac{1}{2}y, \quad y(0) = 1, \quad y'(0) = 0.$$

   Write this second order differential equation as a system of first order differential equations: $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Show that the eigenvalues of $\mathbf{A}$ are given by

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}i \text{ en } \lambda_2 = -\frac{1}{2} - \frac{1}{2}i.$$

(2 pt.)

   (e) Investigate the stability of the trapezoidal rule applied to the system as given in (d). (2 pt.)

[0]**please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. We analyze Lagrangian interpolation. For given points x_0, x_1, ..., x_n, with their respective function values $f(x_0)$, $f(x_1)$, ..., $f(x_n)$, the interpolatory polynomial $p_n(x)$ is given by

$$\begin{aligned} p_n(x) &= \sum_{i=0}^{n} f(x_i) L_i(x), \text{ with} \\ L_i(x) &= \frac{(x - x_0)(\ldots)(x - x_{i-1})(x - x_{i+1})(\ldots)(x - x_n)}{(x_i - x_0)(\ldots)(x_i - x_{i-1})(x_i - x_{i+1})(\ldots)(x_i - x_n)}. \end{aligned} \tag{2}$$

Further, the following measured values have been given in tabular form:

i	x_i	$f(x_i)$
0	-1	3
1	0	2
2	1	5

(a) Give the linear Lagrangian interpolatory polynomial with nodes x_0 and x_1. (1pt.)

(b) Give the quadratic Lagrangian interpolatory polynomial with nodes x_0, x_1 and x_2. (2 pt.)

(c) Calculate $f(0)$ and $f(0.5)$ both by using linear and quadratic Lagrangian interpolation. (2 pt.)

The Newton-Raphson method is based on the following formula:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

(d) Derive the above formula for the Newton-Raphson method. (1.5 pt.)

(e) We are searching the positive zero of $f(x) = e^x - x^3$. Use $p_0 = 3$ as the initial guess and determine p_1 and p_2 by the use of the Newton-Raphson method. (1.5 pt.)

(f) Let p be the solution of $f(p) = 0$. Demonstrate that

$$|p - p_{n+1}| = K|p - p_n|^2, \text{ for } n \to \infty \tag{3}$$

and determine the value of K. (2 pt.)