DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday January 28th 2016, 18:30-21:30

1. We consider the numerical integration of the following initial-value problem

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{1}$$

We are using the *Forward Euler method* to obtain the numerical solution of the initial-value problem (1). This method is given by

$$w_{n+1} = w_n + \Delta t f(t_n, w_n), \tag{2}$$

where Δt and w_n represent the time step size and the numerical solution at time t_n, respectively.

(a) Determine, with appropriate motivation, the order of the local truncation error. (2.5 pt.)

(b) We consider the following second-order initial-value problem

$$\begin{cases} y'' + \varepsilon y' + y = \sin(t), \\ y(0) = 1, \quad y'(0) = 0. \end{cases} \tag{3}$$

Rewrite, with a good motivation, the initial-value problem (3) into the form of a initial-value problem system of first-order differential equations. Take the initial conditions into account. (1 pt.)

We continue with the following system of initial value problems

$$\begin{cases} y_1' = -y_2, \\ y_2' = y_1 + \varepsilon y_2, \end{cases} \tag{4}$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 2$, further ε is a given real-valued constant.

(c) What is the maximum allowable value of Δt for numerical stability if $\varepsilon = 0$? Motivate your answer. (2.5 pt.)

(d) For which values of ε is the given system (analytically) stable? Motivate your answer. (2 pt.)

We investigate numerical stability with the Forward Euler method applied to the given initial-value problem system (4) for general values of ε.

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(e) What is the maximum allowable value of Δt for numerical stability if $-2 \leq \varepsilon < 0$? Motivate your answer. (2 pt.)

2. We consider the boundary-value problem

$$\begin{cases} -y''(x) + (x+1)y(x) = x^3 + x^2 - 2, & 0 < x < 1, \\ y'(0) = 0, \quad y(1) = 1, \end{cases} \tag{5}$$

where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$.

(a) We aim at solving the boundary value problem (5) using finite differences, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size. Give a discretisation method (+proof) where

- the truncation error is of order $\mathcal{O}((\Delta x)^2)$;
- the boundary conditions are taken into account;
- and the discretisation matrix is symmetric.

Use a virtual point for the boundary condition at $x = 0$. (2.5 pt.)

(b) Give the linear system of equations $\mathbf{Aw} = \mathbf{f}$ that results from applying the finite-difference scheme from (a) with three (after processing the virtual points) unknowns (i.e. $\Delta x = 1/3$). (1 pt.)

(c) Since the 3×3 system matrix $\mathbf{A}$ from (b) is symmetric, all eigenvalues are real. Use the Gershgorin circle theorem to estimate the smallest eigenvalue $|\lambda|_{\min}$. From that conclude that the finite-difference scheme from (a) is stable, that is, $\mathbf{A}^{-1}$ exists and there is a constant C such that $\|\mathbf{A}^{-1}\| \leq C$ for $\Delta x \to 0$. (1.5 pt.)

3. We consider the function $f(x) = -x^3 + 6x - 2\frac{7}{8}$.

(a) Define the fixed point iteration by the function $g(x) = \frac{x^3}{6} + \frac{23}{48}$. Show that a fixed point of g is equal to a root of f. Start with $p_0 = 1$ and compute p_1, p_2 and p_3. (2 pt.)

(b) Sketch the fixed point iteration in a figure (plotting $g(p_k)$ as a function of p_k) using the iterates p_0, p_1, p_2 and p_3 computed in (a). (1 pt.)

(c) Show that the fixed point iteration converges for all $p_0 \in [0, 1]$. (2 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html