DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400 WI3097TU)
Thursday August 11th 2016, 18:30-21:30

1. We consider the following method

$$\begin{cases} w_{n+1}^* = w_n + \Delta t f(t_n, w_n) \\ \\ w_{n+1} = w_n + \Delta t \left(a_1 f(t_n, w_n) + a_2 f(t_{n+1}, w_{n+1}^*) \right) \end{cases} \tag{1}$$

for the integration of the **initial value problem** $y' = f(t,y)$, $y(t_0) = y_0$.

   (a) Show that the *local truncation error* of the above method has order $O(\Delta t)$ if $a_1 + a_2 = 1$. Which value for a_1 and a_2 will give a local truncation error of order $O((\Delta t)^2)$? (3 pt.)

   (b) Demonstrate that for *general values of* a_1 and a_2 the *amplification factor* is given by

$$Q(\lambda \Delta t) = 1 + (a_1 + a_2)\lambda \Delta t + a_2 (\lambda \Delta t)^2. \tag{2}$$

   (2 pt.)

   (c) Consider $\lambda < 0$ and $(a_1 + a_2)^2 - 8a_2 < 0$. Derive the *condition for stability*, to be fulfilled by Δt. (2 pt.)

   (d) We consider the following *system of non-linear differential equations*:

$$\begin{array}{l} x_1' = -\sin x_1 + 2x_2 + t, \;\; x_1(0) = 0, \\ x_2' = x_1 - x_2^2, \;\; x_2(0) = 1. \end{array} \tag{3}$$

   Show that the Jacobian of the right-hand side of (3) at $t = 0$ is given by:

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

   (1.5 pt.)

   (e) Choose $a_1 = a_2 = \frac{1}{2}$. For which values of Δt is the method applied to (3) stable at $t = 0$? (1.5 pt.)

2. We are looking for a **difference formula** of the form:

$$Q(h) = \frac{\alpha_0}{h^2} f(0) + \frac{\alpha_{-1}}{h^2} f(-h) + \frac{\alpha_{-2}}{h^2} f(-2h),$$

such that

$$f''(0) - Q(h) = \mathcal{O}(h).$$

   (a) Give the *system of linear equations* that has to be satisfied by α_0, α_{-1} and α_{-2}. (2 pt.)

   (b) The solution of this system is given by $\alpha_0 = 1$, $\alpha_{-1} = -2$ and $\alpha_{-2} = 1$. Give, using these values, an expression for the *truncation error* $f''(0) - Q(h)$. (1 pt.)

please turn over

x	$f(x)$
0	0
$-\frac{1}{4}$	0.0156
$-\frac{1}{2}$	0.1250
$-\frac{3}{4}$	0.4219
-1	1.0000

Table 1: The measured numbers

(c) Using the *Richardson method*, give an estimate of the error $f''(0) - Q(\frac{1}{4})$ using the numbers given in Table 1. (2 pt.)

3. We consider the one-dimensional **convection-diffusion equation** with Dirichlet boundary conditions:

$$\begin{cases} -\epsilon u'' + u' = 1, & 0 < x < 1, \\ u(0) = 0, & u(1) = 0, \end{cases} \tag{4}$$

where $u = u(x)$, $u' = \frac{du}{dx}$ and $u'' = \frac{d^2u}{dx^2}$

(a) Show that

$$u(x) = x - \frac{1 - e^{x/\epsilon}}{1 - e^{1/\epsilon}} \tag{5}$$

is the *exact solution* to the boundary value problem (4). (1 pt.)

(b) We solve the boundary value problem (4) using *central finite differences* for the diffusive term and *upwind finite differences* for the convective term.

For all *interior nodes* x_j the discretization method reads

$$-\epsilon \frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + \frac{w_j - w_{j-1}}{\Delta x} = 1, \text{ for } j \in \{1, \dots, n\}. \tag{6}$$

with $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size.

Give a *discretization method* for the two boundary nodes x_1 and x_n. (1 pt.)

(c) Use a step size of $\Delta x = 1/4$ to derive the system of equations $\mathbf{Aw} = \mathbf{f}$. Take care of the boundary conditions. The system must have three unknowns and three equations, i.e. $\mathbf{A}$ is a 3×3 matrix and $\mathbf{w}$ and $\mathbf{f}$ are 1×3 column vectors.

You do **not** have to solve this system. (2 pt.)

(d) Will the discretization method (6) produce oscillatory solutions? Motivate your answer. (1 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html