DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday April 20th 2017, 18:30-21:30

1. The **Modified Euler Method** to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$\begin{cases} w_{n+1}^* = w_n + \Delta t f(t_n, w_n) \\ w_{n+1} = w_n + \dfrac{\Delta t}{2}\left(f(t_n, w_n) + f(t_{n+1}, w_{n+1}^*)\right), \end{cases} \tag{1}$$

where Δt denotes the time–step and w_n represents the numerical solution at time t_n.

(a) Show that the *local truncation error* of the Modified Euler Method is $\mathcal{O}(\Delta t^2)$. (3 pt.)

(b) The *amplification factor* of the Modified Euler Method is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2}.$$

Derive this amplification factor for the Modified Euler Method. (1 pt.)

(c) Given the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = \sin t, \\ y(0) = 1, \quad \dfrac{dy}{dt}(0) = 2. \end{cases} \tag{2}$$

Show that, this problem can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}. \tag{3}$$

Give also the initial conditions for $x_1(0)$ and $x_2(0)$. (2 pt.)

(d) Perform *one step* with the Modified Euler Method with $\Delta t = 0.1$ and $t_0 = 0$, using the given initial conditions from (2). (2 pt.)

(e) Determine whether the Modified Euler Method, applied to the given initial value problem (2), is stable for $\Delta t = 1$. (2 pt.)

2. In this exercise an estimate is determined for the velocity of a rowing boat. The measured distance of the boat from the starting line are given in the table below.

t (s)	0	10	20
$d(t)$ (m)	0	40	100

(a) Give the **first-order backward difference formul**a and use this to determine an estimate of the velocity for $t = 20$ ($d'(20)$). (1 pt.)

please turn over

(b) We look for a difference formula of the first derivative of d in $2h$ of the form:

$$Q(h) = \frac{\alpha_0}{h} d(0) + \frac{\alpha_1}{h} d(h) + \frac{\alpha_2}{h} d(2h),$$

such that

$$d'(2h) - Q(h) = O(h^2).$$

In the remainder of this exercise we use this formula. Show that the coefficients $\alpha_0,\ \alpha_1$ and α_2 should satisfy the next system:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\ , \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\ , \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\ . \end{array}$$

(2 pt.)

(c) The solution of this system is given by $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$. Give for these values an expression for the truncation error $d'(2h) - Q(h)$. Use this formula to give an estimate of the velocity at $t = 20$. (2 pt.)

3. We derive and use **Newton-Raphson's method** to solve a nonlinear problem.

(a) Given is the *scalar* nonlinear problem:

$$\text{Find } p \in \mathbb{R} \text{ such that } f(p) = 0. \tag{4}$$

Derive Newton–Raphson's formula, given by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \text{ for } n \geq 1 \tag{5}$$

with initial guess p_0 to solve the above problem. (1 pt.)

(b) To prove convergence of Newton-Raphson's method it can be reformulated as fixed-point method for which a rigorous convergence proof exists:

$$g(x) = x - \frac{f(x)}{f'(x)}. \tag{6}$$

One of the prerequisites of the convergence proof is that $g'(x)$ exists and satisfies

$$|g'(x)| \leq k < 1. \tag{7}$$

Show that Newton-Raphson's method applied to $f(x) = \sin(x)$ converges to the root $p = 0$ for any initial guess $p_0 \in (-\pi/4, \pi/4)$. (1 pt.)

(c) Formulate Newton-Raphson's method for the *general* nonlinear problem:

$$\text{Find } \mathbf{p} \in \mathbb{R}^m \text{ such that } \mathbf{f}(\mathbf{p}) = \mathbf{0}. \tag{8}$$

(1.5 pt.)

(d) Perform **one** step of Newton-Raphson's method applied to the following nonlinear problem for w_1 and w_2:

$$\left\{ \begin{array}{r} 18w_1 - 9w_2 + w_1^2 = 0, \\ -9w_1 + 18w_2 + w_2^2 = 9. \end{array} \right. \tag{9}$$

Use $w_1 = w_2 = 0$ as the initial estimate. (1.5 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html