DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday July 6 2017, 18:30-21:30

1. We consider the numerical integration of the following **initial-value problem**

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{1}$$

We are using the *Forward Euler method* to obtain the numerical solution of the initial-value problem (1). This method is given by

$$w_{n+1} = w_n + \Delta t f(t_n, w_n), \tag{2}$$

where Δt and w_n represent the time step size and the numerical solution at time t_n, respectively.

(a) Determine the order of the local truncation error. (2 pt.)

(b) Determine the amplification factor for this method. For which Δt is the method stable if λ is a negative real number? (2 pt.)

(c) We consider the initial value problem:

$$y'' = -y' - \frac{1}{2}y, \;\; y(0) = 1, \;\; y'(0) = 0.$$

Write this second order differential equation as a system of first order differential equations: $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Show that the eigenvalues of $\mathbf{A}$ are given by

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}i \text{ en } \lambda_2 = -\frac{1}{2} - \frac{1}{2}i.$$

(2 pt.)

(d) Do one step with the Forward Euler method applied to the system with $\Delta t = 1$. (2 pt.)

(e) Investigate the stability of the Forward Euler method for this system for a general value of $\Delta t > 0$. (2 pt.)

[0]**please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. We consider the following **iteration process** $x_{n+1} = g(x_n)$, with

$$g(x_n) = x_n + h(x_n)(x_n^3 - 27),$$

where h is a continuous function with $h(x) \neq 0$ for each $x \neq 0$.

(a) If this process converges, to which limit p does it converge? (1 pt.)

(b) Consider the following three possible choices for $h(x)$:

(i) $h_1(x) = -\frac{1}{x^4}$

(ii) $h_2(x) = -\frac{1}{x^2}$

(iii) $h_3(x) = -\frac{1}{3x^2}$

For which choice does the process **not** converge? For which choice is the convergence the fastest? Motivate your answer. (2 pt.)

(c) Find a function $h_4(x)$ such that the 'convergence' factor equals one. (1 pt.)

(d) Let p be the root of a given function f. $\hat{f}$ is the function perturbed by measurement errors. It is given that $|\hat{f}(x) - f(x)| \leq \epsilon_{max}$ for all x. Show that the root $\hat{p}$ from $\hat{f}$ satisfies the following inequality $|\hat{p} - p| \leq \frac{\epsilon_{max}}{|f'(p)|}$. (1 pt.)

3. We consider the **boundary-value problem**

$$\begin{cases} -y''(x) + (x+1)y(x) = x^3 + x^2 - 2, & 0 < x < 1, \\ y'(0) = 0, \quad y(1) = 1, \end{cases} \tag{3}$$

where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$.

(a) We aim at solving the boundary value problem (3) using finite differences, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size.

Give a discretisation method (+proof) where

- the truncation error is of order $\mathcal{O}((\Delta x)^2)$;
- the boundary conditions are taken into account;
- and the discretisation matrix is symmetric.

Use a virtual point for the boundary condition at $x = 0$. (2.5 pt.)

(b) Give the linear system of equations $\mathbf{A}\mathbf{w} = \mathbf{f}$ that results from applying the finite-difference scheme from (a) with three (after processing the virtual points) unknowns (i.e. $\Delta x = 1/3$).

Remark: You do **not** have to solve this linear system of equations. (1 pt.)

(c) Since the 3×3 system matrix $\mathbf{A}$ from (b) is symmetric, all eigenvalues are real. Use the Gershgorin circle theorem to estimate the smallest eigenvalue $|\lambda|_{\min}$. From that conclude that the finite-difference scheme from (a) is stable, that is, $\mathbf{A}^{-1}$ exists and there is a constant C such that $\|\mathbf{A}^{-1}\| \leq C$ for $\Delta x \to 0$. (1.5 pt.)