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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400)
Thursday August 17th 2017, 18:30-21:30

1. We consider the following method

$$\begin{cases} w^*_{n+1} = w_n + \Delta t f(t_n, w_n) \\ \\ w_{n+1} = w_n + \Delta t \left(a_1 f(t_n, w_n) + a_2 f(t_{n+1}, w^*_{n+1})\right) \end{cases} \tag{1}$$

for the integration of the **initial value problem** $y' = f(t, y)$, $y(t_0) = y_0$

(a) Show that the *local truncation error* of the above method has order $O(\Delta t)$ if $a_1 + a_2 = 1$. Which value for a_1 and a_2 will give a local truncation error of order $O((\Delta t)^2)$? (3 pt.)

(b) Demonstrate that for general values of a_1 and a_2 the *amplification factor* is given by

$$Q(\lambda\Delta t) = 1 + (a_1 + a_2)\lambda\Delta t + a_2(\lambda\Delta t)^2. \tag{2}$$

(2 pt.)

(c) Consider $\lambda < 0$ and $(a_1 + a_2)^2 - 8a_2 < 0$. Derive the *condition for stability*, to be fulfilled by Δt. (2 pt.)

We apply the method to the following *system of differential equations*

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2t \end{pmatrix}, \tag{3}$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 0$.

(d) Use $\Delta t = \frac{1}{2}$ to compute w^1 (one time–step) using the method with $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$. (1 pt.)

(e) Is the method with $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$ applied to (3) stable for the choice $\Delta t = \frac{1}{2}$? (motivate your answer) (2 pt.)

2. We analyse **Lagrangian interpolation**. For given points $x_0, x_1, \ldots, x_n$, with their respective function values $f(x_0), f(x_1), \ldots, f(x_n)$, the interpolatory polynomial $L_n(x)$ is given by

$$L_n(x) = \sum_{k=0}^{n} f(x_k) L_{kn}(x), \text{ with}$$
$$L_{kn}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}. \tag{4}$$

please turn over

(a) Give the *linear Lagrangian interpolatory polynomial* $L_1(x)$ with nodes x_0 and x_1. (1 pt.)

(b) Give the *quadratic Lagrangian interpolatory polynomial* $L_2(x)$ with nodes x_0, x_1 and x_2. (2 pt.)

(c) Calculate $L_n(2)$ and $L_n(3)$ both by using linear and quadratic Lagrangian interpolation using the following measured values:

k	x_k	$f(x_k)$
0	1	3
1	3	6
2	4	5

(2 pt.)

3. We are interested in the **numerical integration** of $\int_0^1 y(x)dx$ with $y(x) = x^2$.

(a) Give the *Rectangle Rule* I^R, the corresponding *composed integration rule* $I^R(h)$ and compute the approximate integral $\int_0^1 y(x)dx$ with $h = 1/4$. (1 pt.)

(b) Repeat part (a) for the *Trapezoidal Rule* (I^T and $I^T(h)$) with $h = 1/4$. (1 pt.)

(c) If one approximates $\int_0^1 y(x)dx$, the magnitude of the error of the *composed integration rules* (ε_R and ε_T for the Rectangle and Trapezoidal Rule, respectively) is bounded by

$$\varepsilon_R \leq \frac{h}{2} \max_{x\in[0,1]} |y'(x)|, \qquad \varepsilon_T \leq \frac{h^2}{12} \max_{x\in[0,1]} |y''(x)|. \tag{5}$$

Give *explicit* upper bounds for the error with $y(x) = x^2$. Which method do you recommend if the number of integration points is large? Give a proper motivation. (1 pt.)

(d) Start from the error formula for the *Trapezoidal Rule*

$$\int_0^1 y(x)dx - I^T(h) = c_p h^p$$

and use Richardson's extrapolation to derive the expression

$$\frac{I^T(2h) - I^T(4h)}{I^T(h) - I^T(2h)} = 2^p$$

Compute the numerical approximation order p for $h = 1/4$. (1 pt.)

(e) Derive the relation

$$\int_0^1 y(x)dx - I^T(h) = \frac{Q(h) - Q(2h)}{2^p - 1}$$

and use it to estimate the error of the *Trapezoidal Rule* for $h = 1/4$. (1 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html