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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday April 19th 2018, 18:30-21:30

1. (a) The local truncation error is given by

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t},$$

where $y_{n+1} = y(t_{n+1})$ is the exact solution at time t_{n+1} and z_{n+1} the numerical approximation after one step with $w_n = y_n$ as starting point.
The Taylor series around t_n for y_{n+1} is:

$$y_{n+1} = y_n + \Delta t y_n' + \frac{1}{2}\Delta t^2 y_n'' + \mathcal{O}\left(\Delta t^3\right).$$

The formula for z_{n+1} is

$$z_{n+1} = y_n + \frac{1}{2}\Delta t f\left(t_n, y_n\right) + \frac{1}{2}\Delta t f\left(t_n + \Delta t, y_n + \Delta t f\left(t_n, y_n\right)\right),$$

which has as Taylor series around (t_n, y_n)

$$\begin{aligned} z_{n+1} = y_n &+ \tfrac{1}{2}\Delta t f(t_n, y_n) \\ &+ \tfrac{1}{2}\Delta t \left[f(t_n, y_n) + \Delta t \tfrac{\partial f}{\partial t}(t_n, y_n) \right. \\ &+ \left. \Delta t f(t_n, y_n) \tfrac{\partial f}{\partial y}(t_n, y_n) + \mathcal{O}\left(\Delta t^2\right) \right]. \end{aligned}$$

Using $y_n' = f(t_n, y_n)$ and

$$\begin{aligned} y_n'' &= (y_n')', \\ &= \frac{df}{dt}(t_n, y_n), \\ &= \frac{\partial f}{\partial t}(t_n, y_n) + y_n' \frac{\partial f}{\partial y}(t_n, y_n), \\ &= \frac{\partial f}{\partial t}(t_n, y_n) + f(t_n, y_n) \frac{\partial f}{\partial y}(t_n, y_n), \end{aligned}$$

this becomes

$$z_{n+1} = y_n + \Delta t y_n' + \tfrac{1}{2}\Delta t^2 y_n'' + \mathcal{O}\left(\Delta t^3\right).$$

Hence, this gives

$$y_{n+1} - z_{n+1} \quad = \quad \mathcal{O}\left(\Delta t^3\right),$$

and hence $\tau_{n+1}(\Delta t) = \mathcal{O}(\Delta t^2)$ because

$$\tau_{n+1} = \frac{\mathcal{O}\left(\Delta t^3\right)}{\Delta t} = \mathcal{O}\left(\Delta t^2\right).$$

(b) Consider the test equation $y' = \lambda y$, then it follows that

$$\begin{aligned}
k_1 &= \lambda \Delta t w_n \\
k_2 &= \lambda \Delta t \left(w_n + \lambda \Delta t w_n\right) \\
&= \left(\lambda \Delta t + (\lambda \Delta t)^2\right) w_n \\
w_{n+1} &= w_n + \frac{1}{2}\lambda \Delta t w_n + \frac{1}{2}\left(\lambda \Delta t + (\lambda \Delta t)^2\right) w_n \\
&= \left(1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2\right) w_n.
\end{aligned}$$

Hence the amplification factor is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2.$$

(c) To this extent, we determine the eigenvalues of the matrix A. Subsequently, these eigenvalues are substituted into the amplification factor. The eigenvalues of A are given by $\lambda_1 = -2$, $\lambda_2 = -3$ and $\lambda_3 = -4$, as A is a lower-triangular matrix. These eigenvalues can also be found by deriving the characteristic equation $\det(A - \lambda I) = 0$ and solving for λ.
Substitution of $\lambda_3 = -4$ in the amplification factor gives

$$Q(-4\Delta t) = 1 - 4\Delta t + 8\Delta t^2 \Delta t^2$$

For stability it must hold

$$|Q(-4\Delta t)| \leq 1,$$

which results in the inequalities

$$-1 \leq 1 - 4\Delta t + 8\Delta t^2 \leq 1,$$

as Δt is real.
The left inequality gives:

$$\begin{aligned}
&-1 \leq 1 - 4\Delta t + 8\lambda^2, \\
&\Rightarrow 0 \leq 2 - 4\Delta t + 8\Delta t^2, \\
&\Rightarrow 0 \leq 1 - 2\Delta t + 4\Delta t^2,
\end{aligned}$$

which is satisfied for any Δt as the discriminant $D = (-2)^2 - 4 \cdot 4 \cdot 1 = -12 < 0$ and substitution of $\Delta t = 1$ gives $0 \leq 3$.

The right inequality gives:

$$\begin{aligned} 1 - 4\Delta t + 8\lambda^2 &\leq 1, \\ \Rightarrow -4\Delta t + 8\Delta t^2 &\leq 0, \\ \Rightarrow -4 + 8\Delta t &\leq 0, \\ \Rightarrow 8\Delta t &\leq 4, \\ \Rightarrow \Delta t &\leq \frac{1}{2}. \end{aligned}$$

Because $\lambda_1 > \lambda_2 > \lambda_3$, the stability is determined by $\lambda_3 = -4$. Alternatively, one can show that for $\lambda_1 = -2$ the constraint

$$\Delta t \leq 1,$$

is found and similarly for $\lambda_2 = -3$ the constraint

$$\Delta t \leq \frac{2}{3},$$

is found.

Hence for $\Delta t \leq \frac{1}{2}$,it follows that the method applied to the given system is stable. Note that this conclusion holds for all of the eigenvalues of A.

(d) The given method, applied to the system $\underline{y}' = A\underline{y} + \underline{f}$, gives

$$\begin{cases} \underline{k}_1 &= \Delta t \left(A\underline{w}_n + \underline{f}\,(t_n)\right) \\ \underline{k}_2 &= \Delta t \left(A\,(\underline{w}_n + \underline{k}_1) + \underline{f}\,(t_n + \Delta t)\right) \\ \underline{w}_{n+1} &= \underline{w}_n + \frac{1}{2}\,(\underline{k}_1 + \underline{k}_2) \end{cases}$$

With the initial condition and $\Delta t = \frac{1}{2}$, this gives

$$\begin{cases} \underline{k}_1 &= \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} \\ \underline{k}_2 &= \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} \\ \underline{w}_1 &= \begin{bmatrix} 1/4 \\ 1/4 \\ 0 \end{bmatrix} \end{cases}$$

2. (a) Let $y_j = y(x_j)$, and let $x_n = 1$, hence $h = 1/n$, then

$$
\begin{aligned}
y_{j-1} &= y(x_j - h) = y_j - hy'(x_j) + h^2/2y''(x_j) - h^3/3!y'''(x_j) + O(h^4); \\
y_{j+1} &= y(x_j + h) = y_j + hy'(x_j) + h^2/2y''(x_j) + h^3/3!y'''(x_j) + O(h^4);
\end{aligned} \tag{1}
$$

From the above expressions, it can be seen that

$$
y''(x_j) = \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} + O(h^2), \tag{2}
$$

and hence the error is $O(h^2)$. This gives the following discretisation

$$
\frac{-w_{j-1} + 2w_j - w_{j+1}}{h^2} + w_j = 2e^{x_j}, \qquad \text{for } j = 1 \dots n, \tag{3}
$$

where $x_j = jh$ and $w_j \approx y_j$ is the numerical (finite difference) solution neglecting the error.

(b) Furthermore, we use a virtual gridnode near $x = 1$, $x_{n+1} = 1 + h$, with

$$
0 = y'(1) = \frac{y_{n+1} - y_{n-1}}{2h} + O(h^2), \tag{4}
$$

hence the error is $O(h^2)$. Neglecting the error, and substitution into the discretisation equation $j = n$, yields

$$
\frac{-2w_{n-1} + 2w_n}{h^2} + w_n = 2e. \tag{5}
$$

(c) The boundary condition $y(0) = 2$ at $x = 0$ yields $w_0 = 2$, and the equation for $j = 1$ becomes

$$
\frac{2w_1 - w_2}{h^2} + w_1 = \frac{2}{h^2} + 2e^h. \tag{6}
$$

We get, using $h = 1/3$,

$$
18w_1 - 9w_2 + w_1 = 18 + 2e^{\frac{1}{3}} \tag{7}
$$

For $j = 2$, we obtain

$$
-9w_1 + 18w_2 - 9w_3 + w_2 = 2e^{2/3}. \tag{8}
$$

For $j = 3 = n$, we obtain

$$
-18w_2 + 18w_3 + w_3 = 2e. \tag{9}
$$

Hence, the system of equations reads

$$\begin{cases} 19w_1 - 9w_2 = 18 + 2e^{\frac{1}{3}}, \\ \\ -9w_1 + 19w_2 - 9w_3 = 2e^{2/3}, \\ \\ -18w_2 + 19w_3 = 2e. \end{cases} \tag{10}$$

This linear system does not have a symmetric matrix. Division of the third equation by 2 makes the discretisation matrix symmetric:

$$\begin{cases} 19w_1 - 9w_2 = 18 + 2e^{\frac{1}{3}}, \\ \\ -9w_1 + 19w_2 - 9w_3 = 2e^{2/3}, \\ \\ -9w_2 + \frac{19}{2}w_3 = e. \end{cases} \tag{11}$$

Therefore A and b are given by

$$A = \begin{bmatrix} 19 & -9 & 0 \\ -9 & 19 & -9 \\ 0 & -9 & \frac{19}{2} \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 18 + 2e^{\frac{1}{3}} \\ 2e^{\frac{2}{3}} \\ e \end{bmatrix}.$$

3. (a) The Taylor polynomial $P_1(x)$ of $f(x)$ around b is given by

$$P_1(x) = f(b) + (x-b)f'(b)$$

whereas the truncation error is:

$$f(x) - P_1(x) = \frac{(x-b)^2}{2} f''(\xi), \text{ with } \xi \in [a,b].$$

Integrating $P_1(x)$ gives:

$$\int_a^b P_1(x)dx = \int_a^b f(b) + (x-b)f'(b)dx = (b-a)f(b) - \frac{(a-b)^2}{2} f'(b).$$

Suppose that $M_2 = \max_{\xi \in [a,b]} |f''(\xi)|$. This implies that $|f(x) - P_1(x)| \leq \frac{(x-b)^2}{2} M_2$. Integrating this formula gives:

$$\begin{aligned} \left| \int_a^b f(x)dx - \left((b-a)f(b) - \frac{(a-b)^2}{2} f'(b) \right) \right| &\leq \int_a^b |f(x) - P_1(x)| dx \\ = \int_a^b \frac{(x-b)^2}{2} \left| f''(\xi(x)) \right| dx &\leq \int_a^b \frac{(x-b)^2}{2} M_2 dx \\ &= \frac{(b-a)^3}{6} M_2 \end{aligned}$$

(b) The composite rule $I(h)$ is:

$$I(h) = h \sum_{j=1}^{n} \left(f(x_j) - \frac{h}{2} f'(x_j) \right).$$

For $h = \frac{1}{2}$, $n = 2$, $a = 0$ and $b = 1$ the composite rule becomes

$$\begin{aligned} I\left(\frac{1}{2}\right) &= \frac{1}{2} \sum_{j=1}^{2} \left(f\left(\frac{1}{2}j\right) - \frac{1}{4} f'\left(\frac{1}{2}j\right) \right) \\ &= \frac{1}{2} \left(f\left(\frac{1}{2}\right) - \frac{1}{4} f'\left(\frac{1}{2}\right) + f(1) - \frac{1}{4} f'(1) \right). \end{aligned}$$

Using $f(x) = x^3$ and $f'(x) = 3x^2$ gives:

$$\begin{aligned} I\left(\frac{1}{2}\right) &= \frac{1}{2} \left(\frac{1}{8} - \frac{3}{16} + 1 - \frac{3}{4} \right) \\ &= \frac{1}{2} \frac{6}{32} \\ &= \frac{3}{32} \\ &= 0.09375. \end{aligned}$$

The difference with the exact answer $\int_0^1 x^3 dx = \frac{1}{4}$ is

$$\int_0^1 x^3 dx - I\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{3}{32} = \frac{5}{32} = 0.15625.$$

(c) For the comparison we provide the following table for the composite methods:

Aspect	**New method**	**Trapezoidal rule**
Number of function evaluations	$2n$	$n+1$
Truncation error	$\frac{(b-a)h^2}{6} M$	$\frac{(b-a)h^2}{12} M$
Rounding errors from	f and f'	f

where $M = \max_{x \in [a,b]} |y''(x)|$.

Comparing the two methods, one can conclude:

- the new method has a worse behaviour with respect to rounding errors, because rounding errors of f' also play a role;
- the new method costs $n-1$ function evaluations more than the Trapezoidal rule;
- the truncation error of the new method is two times as large as the truncation error of the Trapezoidal rule.

Conclusion: the new method is worse than the Trapezoidal rule, so the preference should be the Trapezoidal rule.