DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400)
Thursday July 5th 2018, 13:30-16:30

Number of questions: This is an exam with 10 open questions, subdivided in 3 main questions.
Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.
Tools Only a non-graphical calculator is permitted. All other tools are not permitted.
Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/20$, where P is the number of points earned.

1. A method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$w_{n+1} = w_n + (1-\theta)\Delta t f(t_n, w_n) + \theta \Delta t f(t_{n+1}, w_{n+1}),$$

where Δt denotes the time–step, w_n represents the numerical solution at time t_n and $0 \leq \theta \leq 1$.

(a) The *amplification factor* of this method is given by

$$Q(\lambda\Delta t) = \frac{1 + (1-\theta)\lambda\Delta t}{1 - \theta\lambda\Delta t}.$$

Derive this amplification factor for the given method. ($1\frac{1}{2}$ pt.)

(b) Show that the *local truncation error* of the given method is $\mathcal{O}(\Delta t)$ in general for the test equation $y' = \lambda y$. Also determine for which value of θ the method is $\mathcal{O}(\Delta t^2)$. ($3\frac{1}{2}$ pt.)
Hint: $e^x = 1 + x + \frac{1}{2}x^2 + \mathcal{O}(x^3)$.
Hint: $\frac{1}{1-x} = 1 + x + x^2 + \mathcal{O}(x^3)$ for $|x| < 1$.

(c) Take $\theta = \frac{1}{2}$. Given is the initial value problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (1)$$

with initial conditions $x_1(0) = 1, x_2(0) = 0$.
Is the given method applied to this initial value problem *stable* for $\Delta t = 1$? ($3\frac{1}{2}$ pt.)

(d) Perform *one step* with the given method with $\Delta t = 1$, $\theta = \frac{1}{2}$ and $t_0 = 0$ for the initial value problem (1) and the given initial conditions. ($1\frac{1}{2}$ pt.)

please turn over

2. We will analyse *Lagrange interpolation.* For given points $x_0, x_1, \ldots, x_n$ and their respective function values $f(x_0), f(x_1), \ldots, f(x_n)$, the interpolating polynomial $L_n(x)$ is given by

$$L_n(x) = \sum_{k=0}^{n} f(x_k) L_{kn}(x), \quad \text{with}$$
$$L_{kn}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}.$$

(a) *Determine* $\hat{L}_2(2)$ (the perturbed version of $L_2(2)$) given the following *measured* values:

k	x_k	$\hat{f}(x_k)$
0	1	3
1	3	6
2	4	5

(3 pt.)

(b) Given is that we know

$$\left|f(x) - \hat{f}(x)\right| \leq \epsilon,$$
$$|f'''(x)| \leq \delta,$$

and

$$f(x) - L_n(x) = \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\zeta(x)),$$

for $x \in [1,4]$. Determine an *upper bound* for the error $\left|f(2) - \hat{L}_2(2)\right|$. (2 pt.)

3. We want to find a root of the function $f(x) = -x^3 + 6x - \frac{23}{8}$.

(a) We choose to use the *fixed point iteration* $p_{n+1} = g(p_n)$, with $g(x) = \frac{x^3}{6} + \frac{23}{48}$ to find a root. *Show* that a fixed point of $g(x)$ is also a root of $f(x)$. (1 pt.)

(b) We start the fixed point iteration at $p_0 = 1$. *Calculate* p_1, p_2 and p_3 exact to four decimals and *sketch* the fixed point iteration in a figure. (2 pt.)

(c) Show that the chosen fixed point iteration *converges* for all $p_0 \in [0,1]$. (2 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html