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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400)

Thursday June 23 2022, 13:30-16:30

1. (a) The local truncation error is given by

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}, \tag{1}$$

in which we determine y_{n+1} by the use of Taylor expansions around t_n:

$$y_{n+1} = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + \mathcal{O}(\Delta t^3). \tag{2}$$

We bear in mind that

$$y'(t_n) = f(t_n, y_n)$$

$$\begin{aligned} y''(t_n) &= \frac{df(t_n,y_n)}{dt} = \frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y} y'(t_n) \\ &= \frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y} f(t_n,y_n). \end{aligned}$$

Hence

$$y_{n+1} = y_n + \Delta t y'(t_n) + \frac{\Delta t^2}{2}\left(\frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y} f(t_n,y_n)\right) + \mathcal{O}(\Delta t^3). \tag{3}$$

After substitution of the predictor $z_{n+1}^* = y_n + \Delta t f(t_n, y_n)$ into the corrector, and after using a Taylor expansion around (t_n, y_n), we obtain for z_{n+1}:

$$\begin{aligned} z_{n+1} &= y_n + \frac{\Delta t}{2}\left(f(t_n,y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n,y_n))\right) \\ &= y_n + \frac{\Delta t}{2}\left(2f(t_n,y_n) + \Delta t\left(\frac{\partial f(t_n,y_n)}{\partial t} + f(t_n,y_n)\frac{\partial f(t_n,y_n)}{\partial y}\right) + \mathcal{O}(\Delta t^2)\right). \end{aligned}$$

Herewith, one obtains

$$y_{n+1} - z_{n+1} = \mathcal{O}(\Delta t^3), \text{ and hence } \tau_{n+1}(\Delta t) = \frac{\mathcal{O}(\Delta t^3)}{\Delta t} = \mathcal{O}(\Delta t^2). \tag{4}$$

(b) Let $x_1 = y$ and $x_2 = y'$, then $y'' = x_2'$, and hence

$$\begin{aligned} &x_2' + 4x_1 + 4x_2 = \cos(\pi t), \\ &x_1' = x_2. \end{aligned} \tag{5}$$

We write this as

$$\begin{cases} x_1' &= x_2, \\ x_2' &= -4x_1 - 4x_2 + \cos(\pi t). \end{cases} \tag{6}$$

Finally, this is represented in the following matrix-vector form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(\pi t) \end{pmatrix}. \tag{7}$$

In which, we have the following matrix $A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$ and $\underline{f} = \begin{pmatrix} 0 \\ \cos(\pi t) \end{pmatrix}$. The initial conditions are defined by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) *Note: Every miscalculation in the calculation of $\underline{w}_1^*$ gives a subtraction of 1/4 point, with at most 1/2 point being subtracted.*

Note: The calculation of $\underline{w}_1$ must be consistent with the value for $\underline{w}_1^*$. If not, 1 point is subtracted.

Note: Every miscalculation in the calculation of $\underline{w}_1$ gives a subtraction of 1/4 point, with at most 1 point being subtracted.

Application of the integration method to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{aligned} \underline{w}_1^* &= \underline{w}_0 + \Delta t \left(A\underline{w}_0 + \underline{f}_0 \right), \\ \underline{w}_1 &= \underline{w}_0 + \tfrac{\Delta t}{2} \left(A\underline{w}_0 + f_0 + A\underline{w}_1^* + \underline{f}_1 \right). \end{aligned} \tag{8}$$

With the initial condition $\underline{w}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Delta t = 0.5$, this gives the following result for the predictor

$$\underline{w}_1^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.5 \left(\begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}. \tag{9}$$

The corrector is calculated as follows

$$\begin{aligned} \underline{w}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.25 \left(\begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0.625 \\ -0.25 \end{pmatrix} \end{aligned}$$

(d) Consider the test equation $y' = \lambda y$, then one gets

$$\begin{aligned} w_{n+1}^* &= w_n + \Delta t \lambda w_n = (1 + \Delta t \lambda) w_n, \\ w_{n+1} &= w_n + \frac{\Delta t}{2} (\lambda w_n + \lambda w_{n+1}^*) \\ &= w_n + \frac{\Delta t}{2} (\lambda w_n + \lambda (w_n + \Delta t \lambda w_n)) \\ &= \left(1 + \Delta t \lambda + \frac{(\Delta t \lambda)^2}{2} \right) w_n. \end{aligned}$$

Hence the amplification factor is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2}. \tag{10}$$

(e) *Note: Every miscalculation in the calculation of $|Q(\lambda_1 \Delta t)|^2$ gives a subtraction of 1/4 point, with at most 1/2 point being subtracted.*

Note: The calculation of $|Q(\lambda_1 \Delta t)|^2$ *must be consistent with the eigenvalues found. If not, 1/2 point is subtracted.*

First, we determine the eigenvalues of the matrix A. Subsequently, the eigenvalues are substituted into the amplification factor.

The eigenvalues of the matrix A are given by $\lambda_1 = -2$ and $\lambda_2 = -2$.

Since both eigenvalues are the same it is sufficient to check if $|Q(\lambda_1 \Delta t)| \leq 1$. Since $Q(\lambda_1 \Delta t) = 1 + \lambda_1 \Delta t + \frac{1}{2}(\lambda_1 \Delta t)^2$ we have to check that $|1 - 2\Delta t + 2(\Delta t)^2| \leq 1$. This leads to

$$-1 \leq 1 - 2\Delta t + 2(\Delta t)^2 \leq 1.$$

We start with the left inequality:

$$-1 \leq 1 - 2\Delta t + 2(\Delta t)^2$$

This can be written as

$$0 \leq 2 - 2\Delta t + 2(\Delta t)^2$$

This is a second order polynomial. Since the discriminant $(-2)^2 - 4 \times 2 \times 2$ is negative there are no real roots. The inequality holds for $\Delta t = 0$ so it holds for all Δt-values.

For the right inequality we have:

$$1 - 2\Delta t + 2(\Delta t)^2 \leq 1.$$

This is equivalent to

$$-2\Delta t + 2(\Delta t)^2 \leq 0.$$

Dividing

$$2(\Delta t)^2 \leq -2\Delta t$$

by $2\Delta t$ leads to

$$\Delta t \leq 1.$$

So the method is stable for all $\Delta t \leq 1$.

2. (a) The first order backward difference formula for the first derivative is given by

$$d'(t) \approx \frac{d(t) - d(t-h)}{h}.$$

Using $t = 20$, and $h = 10$ the approximation of the velocity is

$$\frac{d(20) - d(10)}{10} = \frac{100 - 40}{10} = 6 \text{ (m/s)}.$$

(b) Taylor polynomials are:

$$\begin{aligned} d(0) &= d(2h) - 2hd'(2h) + 2h^2 d''(2h) - \frac{(2h)^3}{6} d'''(\xi_0) \,, \\ d(h) &= d(2h) - hd'(2h) + \frac{h^2}{2} d''(2h) - \frac{h^3}{6} d'''(\xi_1) \,, \\ d(2h) &= d(2h). \end{aligned}$$

We know that $Q(h) = \frac{\alpha_0}{h} d(0) + \frac{\alpha_1}{h} d(h) + \frac{\alpha_2}{h} d(2h)$, which should be equal to $d'(2h) + O(h^2)$. This leads to the following conditions:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0 \,, \\ -2\alpha_0 & - & \alpha_1 & & & = & 1 \,, \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0 \,. \end{array}$$

(c) The truncation error follows from the Taylor polynomials:

$$d'(2h) - Q(h) = d'(2h) - \frac{d(0) - 4d(h) + 3d(2h)}{2h} = \frac{\frac{8h^3}{6} d'''(\xi_0) - 4(\frac{h^3}{6} d'''(\xi_1))}{2h} = O(h^2).$$

(d) Using the new formula with $h = 10$ we obtain the estimate:

$$\frac{d(0) - 4d(10) + 3d(20)}{20} = \frac{0 - 4 \times 40 + 3 \times 100}{20} = 7 \text{ (m/s)}.$$

3. (a) **Newton-Raphson's method** is an iterative method to find $p \in \mathbb{R}$ such that $f(p) = 0$. Suppose $f \in C^2[a,b]$. Let $\bar{x} \in [a,b]$ be an approximation of the root p such that $f'(\bar{x}) \neq 0$, and suppose that $|p - \bar{x}|$ is small. Consider the first-degree Taylor polynomial about $\bar{x}$:

$$f(x) = f(\bar{x}) + (x - \bar{x})f'(\bar{x}) + \frac{(x-\bar{x})^2}{2} f''(\xi(x)), \tag{11}$$

in which $\xi(x)$ between x and $\bar{x}$. Using that $f(p) = 0$, equation (11) yields

$$0 = f(\bar{x}) + (p - \bar{x})f'(\bar{x}) + \frac{(p-\bar{x})^2}{2} f''(\xi(x)).$$

Because $|p - \bar{x}|$ is small, $(p - \bar{x})^2$ can be neglected, such that

$$0 \approx f(\bar{x}) + (p - \bar{x})f'(\bar{x}).$$

Note that the right-hand side is the formula for the tangent in $(\bar{x}, f(\bar{x}))$. Solving for p yields

$$p \approx \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})}.$$

This motivates the Newton-Raphson method, that starts with an approximation p_0 and generates a sequence $\{p_n\}$ by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for} \quad n \geq 1.$$

Remark 1 *One can also give a graphical derivation following Figure 4.2 from the book.*

(b) It follows from the linearization of the function $\mathbf{f}$ about the iterate $\mathbf{x}_{n-1}$ that

$$f_1(\mathbf{p}) \approx f_1(\mathbf{p}^{(n-1)}) + \frac{\partial f_1}{\partial p_1}(\mathbf{p}^{(n-1)})(p_1 - p_1^{(n-1)}) + \ldots + \frac{\partial f_1}{\partial p_m}(\mathbf{p}^{(n-1)})(p_m - p_m^{(n-1)}),$$

$$\vdots$$

$$f_m(\mathbf{p}) \approx f_m(\mathbf{p}^{(n-1)}) + \frac{\partial f_m}{\partial p_1}(\mathbf{p}^{(n-1)})(p_1 - p_1^{(n-1)}) + \ldots + \frac{\partial f_m}{\partial p_m}(\mathbf{p}^{(n-1)})(p_m - p_m^{(n-1)}).$$

Defining the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ by

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial f_1}{\partial x_m}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial f_m}{\partial x_m}(\mathbf{x}) \end{pmatrix},$$

the linearization can be written in the more compact form

$$\mathbf{f}(\mathbf{p}) \approx \mathbf{f}(\mathbf{p}^{(n-1)}) + \mathbf{J}(\mathbf{p}^{(n-1)})(\mathbf{p} - \mathbf{p}^{(n-1)}).$$

The next iterate, $\mathbf{p}^{(n)}$, is obtained by setting the linearization equal to zero:

$$\mathbf{f}(\mathbf{p}^{(n-1)}) + \mathbf{J}(\mathbf{p}^{(n-1)})(\mathbf{p}^{(n)} - \mathbf{p}^{(n-1)}) = 0, \tag{12}$$

which can be rewritten as

$$\mathbf{J}(\mathbf{p}^{(n-1)})\mathbf{s}^{(n)} = -\mathbf{f}(\mathbf{p}^{(n-1)}), \tag{13}$$

where $\mathbf{s}^{(n)} = \mathbf{p}^{(n)} - \mathbf{p}^{(n-1)}$. The new approximation equals $\mathbf{p}^{(n)} = \mathbf{p}^{(n-1)} + \mathbf{s}^{(n)}$. Finally, Newton-Raphson's formula for general nonlinear problems reads:

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n-1)} - \mathbf{J}^{-1}(\mathbf{p}^{(n-1)})\mathbf{f}(\mathbf{p}^{(n-1)}). \tag{14}$$

(c) First, we rewrite the system into the form

$$\begin{aligned} f_1(w_1, w_2) &= 0, \\ f_2(w_1, w_2) &= 0, \end{aligned} \tag{15}$$

by setting

$$\begin{aligned} f_1(w_1, w_2) &:= 18w_1 - 9w_2 + (w_1)^2, \\ f_2(w_1, w_2) &:= -9w_1 + 18w_2 + (w_2)^2 - 9. \end{aligned} \tag{16}$$

We denote the Jacobi-matrix by $J(w_1, w_2)$. At the first step we compute

$$\underline{w}^{(1)} = \underline{w}^{(0)} - J(\underline{w}^{(0)})^{-1} F(\underline{w}^{(0)}), \tag{17}$$

where $\underline{w} = [w_1 \; w_2]^T$. Note that

$$J(\underline{w}^{(0)}) = \begin{pmatrix} 18 + 2w_1^{(0)} & -9 \\ -9 & 18 + 2w_2^{(0)} \end{pmatrix}. \tag{18}$$

Using $w_1^{(0)} = w_2^{(0)} = 0$ we obtain:

$$J(\underline{w}^{(0)}) = \begin{pmatrix} 18 & -9 \\ -9 & 18 \end{pmatrix}. \tag{19}$$

This implies that

$$J(\underline{w}^{(0)})^{-1} = \frac{1}{18^2 - 81} \begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix}. \tag{20}$$

Furthermore

$$F(\underline{w}^{(0)}) = \begin{pmatrix} 0 \\ -9 \end{pmatrix}, \tag{21}$$

so

$$\underline{w}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{18^2 - 81} \begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}. \tag{22}$$