DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3197Minor & AESB2210-18)
February 3rd, 2023, 13:30 - 15:30

Number of questions: This is an exam with 9 open questions, subdivided in 3 main questions.
Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will give less or no points.
Electronic tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.
Notes, book and formula sheets Notes, books and formula sheets are not permitted.
Assessment In total 20 points can be earned. The final grade is given by $\max\{1, P/2\}$ rounded to one decimal, where P is the number of points earned.

1. Consider the following system of differential equations

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \cos(\pi t) \\ 0 \end{pmatrix}, \tag{A}$$

combined with the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$ into an initial value problem.

For this initial value problem we use the following implicit numerical time integration method:

$$\begin{cases} k_1 = f(t_n + \frac{1}{2}\Delta t, w_n + \frac{1}{2}\Delta t k_1) \\ w_{n+1} = w_n + \Delta t k_1. \end{cases} \tag{B}$$

Here Δt denotes the time step and w_n represents the numerical approximation of $y(t_n)$ after n time steps.

(a) Show that the amplification factor $Q(\lambda\Delta t)$ of the above integration method (B) is given by: ($2\frac{1}{2}$ pt.)

$$Q(\lambda\Delta t) = \frac{1 + \frac{1}{2}\lambda\Delta t}{1 - \frac{1}{2}\lambda\Delta t}.$$

(b) Show that the local truncation error of the above time integration method (B) is of the order $\mathcal{O}(\Delta t^2)$ for the test equation $y' = \lambda y$. ($2\frac{1}{2}$ pt.)
Hint 1: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \mathcal{O}(x^4)$
Hint 2: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \mathcal{O}(x^4)$

(c) Determine for which time steps $\Delta t > 0$, the integration method (B), applied to the system (A), is stable. ($3\frac{1}{2}$ pt.)

(d) Calculate one step with the time integration method (B), in which $\Delta t = 1$ and $t_0 = 0$, applied to (A) and use the given initial conditions. (2 pt.)

please turn over

2. We consider the following boundary-value problem:

$$\begin{cases} -y''(x) + 13y(x) &= \frac{1}{2+x}, \quad x \in [0,1], \\ y(0) &= 1, \\ y'(1) &= 0. \end{cases} \tag{C}$$

In this exercise we try to approximate the exact solution with a numerical method.

We solve the boundary value problem (C) using central finite differences with a local truncation error of $\mathcal{O}(\Delta x^2)$, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size. After discretization we obtain the following formulas:

$$-\frac{w_2 - 2w_1}{\Delta x^2} + 13w_1 = \frac{1}{2+x_1} + \frac{1}{\Delta x^2},$$

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} + 13w_j = \frac{1}{2+x_j}, \qquad \text{for } j \in \{2, \ldots, n\},$$

$$-\frac{-w_{n+1} + w_n}{\Delta x^2} + \frac{13}{2}w_{n+1} = \frac{1}{6}.$$

(a) Give (with arguments) the derivation of this scheme. (3 pt.)

(b) Choose $\Delta x = 1/3$ and derive the system of equations resulting from this choice. Furthermore, rewrite this system to the form $A\mathbf{w} = \mathbf{b}$ with $\mathbf{w} = [w_1, \ldots, w_{n+1}]^T$. Explicitly state A and $\mathbf{b}$ in your answer. (1 pt.)

3. Given is that the Trapezoidal rule satisfies

$$\left| \int_{x_L}^{x_R} f(x)\,\mathrm{d}x - \frac{x_R - x_L}{2}\left(f(x_L) + f(x_R)\right) \right| \leq \frac{1}{12} m_2 (x_R - x_L)^3,$$

where $m_2 = \max_{x_L \leq x \leq x_R} |f''(x)|$.

We want to approximate the integral $\int_a^b f(x)\,\mathrm{d}x$ using the composite Trapezoidal rule I_T.

(a) Give the formula for the composite Trapezoidal I_T with stepsize $h = \frac{b-a}{n}$ that approximates $\int_a^b f(x)\,\mathrm{d}x$ and show that the composite Trapezoidal rule I_T satisfies

$$\left| \int_a^b f(x)\,\mathrm{d}x - I_T \right| \leq \frac{1}{12} M_2 (b-a) h^2,$$

where $M_2 = \max_{a \leq x \leq b} |f''(x)|$. ($2\frac{1}{2}$ pt.)

(b) Approximate $\int_0^4 x^2\,\mathrm{d}x$ with the composite Trapezoidal rule with $h = 1$. (1 pt.)

(c) Give an appropriate upper bound for the absolute value of the error in the approximation in (b) and compare this error with the absolute value of the exact error. (2 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html