Extra exercises

Section 6.2

1. If D=$\{(t,y)|0 \leq t \leq 1, -2 \leq y \leq 5\}$ and $f(t,y) = (t+1)|y|$. Is $f(t,y)$ Lipschitz-continuous in the variable y?

2. (a) Let D=$\{(t,y)|0 \leq t \leq 2, -\infty < y < \infty\}$. Is $f(t,y) = y - t^2 + 1$ Lipschitz continuous in the variable y?

   (b) Is the initial value problem

   $$\frac{dy}{dt} = y - t^2 + 1, 0 \leq t \leq 2, y(0) = 1$$

   well posed?

Section 6.3

1. (a) What's the general formula for Euler Forwards, with equidistant stepsize h?

   (b) Compose the formula for equidistant Euler Forwards approximation for the initial value problem: $y' = y - t^2 + 1$, $0 \leqslant t \leqslant 2$, $y(0) = 2$. Use $n = 10$.

   (c) Calculate 3 steps and then compare with the exact solution, $y(t) = (t+1)^2 + e^t$.

2. (a) What's the general formula for Euler Backwards, with equidistant stepsize h?

   (b) Compose the formula for equidistant Euler Backwards approximation for the initial value problem: $y' = y - t^2 + 1$, $0 \leqslant t \leqslant 2$, $y(0) = 2$. Use N=10

   (c) Calculate 3 steps and then compare with the exact solution, $y(t) = (t+1)^2 + e^t$.

3. (a) What's the general formula for Trapezoidal rule, with equidistant stepsize h?

   (b) Compose the formula for equidistant Trapezoidal rule approximation for the initial value problem: $y' = y - t^2 + 1$, $0 \leqslant t \leqslant 2$, $y(0) = 2$. Use $n = 10$.

   (c) Calculate 3 steps and then compare with the exact solution, $y(t) = (t+1)^2 + e^t$.

   (d) Compare the error of the Trapezoidal rule with those of the Euler methods.

4. (a) What's the general formula for Modified Euler, with equidistant stepsize h?

   (b) Compose the formula for equidistant Modified Euler approximation for the initial value problem: $y' = y - t^2 + 1$, $0 \leqslant t \leqslant 2$, $y(0) = 2$. Use $n = 10$.

   (c) Calculate 3 steps and then compare with the exact solution, $y(t) = (t+1)^2 + e^t$.

Section 6.4

1. (a) Apply Euler Forwards to the test equation.

   (b) What is the amplification factor?

2. (a) Apply Euler Backwards to the test equation.

   (b) What is the amplification factor?

3. (a) Apply the Trapezium-rule to the test equation.

   - (b) What is the amplification factor?

4. (a) Apply Modified Euler to the test equation.
   - (b) What is the amplification factor?

5. (a) Given: $y' = -5y^2 + 4, \qquad y(0) = 0.$
   Compute λ in the point $(\hat{t}, \hat{y})$, when $\hat{y} \geqslant 0$ (Hint: use the theory mentioned in the stability of a general initial-value problem)
   - (b) Is the previously given initial value problem stable?
   - (c) When is Euler Forward stable?

6. (a) Given: $y' = -20y^2 + 4, \qquad y(0) = 3.$
   Compute λ in the point $(\hat{t}, \hat{y})$, when $\hat{y} \geqslant 0$ (Hint: use the theory mentioned in the stability of a general initial-value problem)
   - (b) Is the previously given initial value problem stable?
   - (c) When is Euler Backward stable?

7. What is the exact solution for the test equation?

8. What is the order of τ_{j+1} for Euler Forward?

9. What is the order of τ_{j+1} for Euler Backward?

Section 6.5

1. (a) What is the general formula for Runge Kutta 4?
   - (b) Given the initial value problem $y' = y - t^2 + 1, 0 \leq t \leq 2, y(0) = 2$. Do 3 steps with RK4 with $h = 0.2$ and compare the solution of step 3 with the exact solution $y = (t+1)^2 + e^t$.

2. (a) Given the initial value problem $y' = -4y^2 + 5, y(0) = 5$
   Compute λ as function of y.
   - (b) Is the initial value problem stable for $\hat{y} > 0$?
   - (c) When is Runge Kutta 4 stable?

Section 6.6

1. (a) Given the initial value problem $y' = -4y^2 + 5, y(0) = 2$.
   Do one step with Euler Forwards with stepsize $h = 0.1$.
   - (b) Do two steps with Euler Forwards with stepsize $h = 0.05$.
   - (c) What is the order of Euler Forwards?
     So what is the value of p?
   - (d) Make an approximation of the error made.

2. (a) Given the initial value problem $y' = y - t^2 + 1, y(0) = 2$.
   Do one step with Modified Euler with stepwidth $h = 0.1$.
   - (b) Do two steps with Modified Euler with $h = 0.05$.
   - (c) What is the order of Modified Euler?
   - (d) Make an approximation of the error made.

Section 6.7

1. (a) Give, in vectorform, the general formula for Euler Forwards.

   (b) Do a step with Euler Forwards, with stepwidth $h = 0.1$ for the system:
   $$\begin{array}{rcl} u_1' & = & -4u_1 - 2u_2 + \cos(t) + 4\sin(t) \\ u_2' & = & 3u_1 + u_2 - 3\sin(t) \end{array}$$
   with initial values $u_1(0) = 0$ and $u_2(0) = -1$.

   (c) Compare the answer with the exact solution
   $$\begin{array}{rcl} u_1(t) & = & 2e^{-t} - 2e^{-2t} + \sin(t) \\ u_2(t) & = & -3e^{-t} + 2e^{-2t}. \end{array}$$

2. (a) Write the equation $y'' - 2y' + y = te^t - t$, with initial values $y(0) = 1$ and $y'(0) = 2$ as a system of first order differential equations.

   (b) Do one step with Euler Forwards with stepwidth $h = 0.1$

   (c) Compute the error wiht the exact solution $y(t) = 3e^t - 2 - t + \frac{1}{6}t^3e^t$.

Answers of the extra exercises

Section 6.2

1. $0 \le t \le 1, -2 \le y \le 5$
   By definition, we know that $f(t,y)$ is Lipschitz continuous in y if there exists $L > 0$ such that: $|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$
   $$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= |(t+1)|y_1| - (t+1)|y_2|| \\ &= |t+1|||y_1| - |y_2|| \le 2|y_1 - y_2| \end{aligned}$$
   So $f(t,y)$ is Lipschitz continuous with $L = 2$.

2. (a) If $f(t,y)$ is Lipschitz continuous, we must have $|\frac{\partial f}{\partial y}(t,y)| \le L$ for all $(t,y) \in D$.
   $\left|\frac{\partial(y-t^2+1)}{\partial y}\right| = |1| = 1$
   So the function is Lipschitz-continuous with $L = 1$.

   (b) (Theorem 6.2.21) $f(t,y) = y - t^2 + 1$ is continuous on D and Lipschitz-continuous in y, so the problem is well posed.

Section 6.3

1. (a) $w_{n+1} = w_n + hf(t_n, w_n)$

   (b) $h = \frac{2-0}{10} = 0.2$, $t_n = 0.2n$, $w_0 = 2$
   $$\begin{aligned} w_{n+1} &= w_n + hf(t_n, w_n) \\ &= w_n + 0.2(w_n - t_n^2 + 1) \\ &= w_n + 0.2(w_n - 0.04n^2 + 1) \\ &= 1.2w_n - 0.008n^2 + 0.2 \end{aligned}$$

   (c) We use the formula from 1b
   $$\begin{aligned} w_1 \approx u(t_1) &= 1.2 \cdot w_0 - 0.008 \cdot 0^2 + 0.2 \\ &= 1.2 \cdot 2 + 0.2 \\ &= 2.6 \\ w_2 \approx u(t_2) &= 1.2 \cdot w_1 - 0.008 \cdot 1^2 + 0.2 \\ &= 1.2 \cdot 2.6 - 0.008 + 0.2 \\ &= 3.312 \\ w_3 \approx u(t_3) &= 1.2 \cdot w_2 - 0.008 \cdot 2^2 + 0.2 \\ &= 1.2 \cdot 3.312 - 0.032 + 0.2 \\ &= 4.1424 \end{aligned}$$
   3 steps $\Rightarrow t_3 = 0.2 \cdot 3 = 0.6$
   $y(0.6) = (0.6+1)^2 + e^{0.6} = 2.56 + e^{0.6} = 4.3821188$
   The absolute error is equal to: $|y(0.6) - w_3| = |4.3821188 - 4.1424| = 0.2397188$

2. (a) $w_{n+1} = w_n + hf(t_{n+1}, w_{n+1})$

   (b) $h = \frac{2-0}{10} = 0.2$, $t_n = 0.2n$, $w_0 = 2$
   $$\begin{aligned} w_{n+1} &= w_n + 0.2(w_{n+1} - t_{n+1}^2 + 1) \\ &= w_n + 0.2(w_{n+1} - 0.04(n+1)^2 + 1) \\ &= w_n + 0.2(w_{n+1} - 0.04(n^2 + 2n + 1) + 1) \\ &= w_n + 0.2w_{n+1} - 0.008n^2 - 0.016n - 0.008 + 2 \\ &= w_n - 0.2w_{n+1} - 0.008n^2 - 0.016n + 0.192 \\ 0.8w_{n+1} &= w_n - 0.008n^2 - 0.016n + 0.192 \\ w_{n+1} &= 1.25w_n - 0.01n^2 - 0.02n + 0.24 \end{aligned}$$

(c) We use the formula from 2b

$$\begin{aligned}
w_0 &= 2\\
w_1 &= 1.25 \cdot w_0 - 0.01 \cdot 0^2 - 0.02 \cdot 0 + 0.24\\
&= 1.25 \cdot 2 + 0.24\\
&= 2.74\\
w_2 &= 1.25 \cdot w_1 - 0.01 \cdot 1^2 - 0.02 \cdot 1 + 0.24\\
&= 1.25 \cdot 2.74 - 0.01 - 0.02 + 0.24\\
&= 3.635\\
w_3 &= 1.25 \cdot w_2 - 0.01 \cdot 2^2 - 0.02 \cdot 2 + 0.24\\
&= 1.25 \cdot 3.635 - 0.04 - 0.04 + 0.24\\
&= 4.70375
\end{aligned}$$

3 steps $\Rightarrow t_3 = 0.2 \cdot 3 = 0.6$

$y(0.6) = (0.6+1)^2 + e^{0.6} = 2.56 + e^{0.6} = 4.3821188$

The absolute error is equal to: $|y(0.6) - w_3| = |4.3821188 - 4.70375| = 0.3216312$

3. (a) $w_{n+1} = w_n + \frac{h}{2}[f(t_n, w_n) + f(t_{n+1}, w_{n+1})]$

(b) $h = \frac{2-0}{10} = 0.2$, $t_n = 0.2n$, $w_0 = 2$

$$\begin{aligned}
w_{n+1} &= w_n + \tfrac{h}{2}[(w_n - t_n^2 + 1) + (w_{n+1} - t_{n+1}^2 + 1)]\\
&= w_n + \tfrac{0.2}{2}(w_n - 0.04n^2 + 1 + w_{n+1} - 0.04(n+1)^2 + 1)\\
&= w_n + 0.1w_n - 0.004n^2 + 0.1 + 0.1(w_{n+1} - 0.004(n^2 + 2n + 1) + 0.1\\
&= w_n + 0.1w_n - 0.004n^2 + 0.1 + 0.1w_{n+1} - 0.004n^2 - 0.008n - 0.004 + 0.1\\
&= 1.1w_n + 0.1w_{n+1} - 0.008n^2 - 0.008n + 0.196\\
0.9w_{n+1} &= 1.1w_n - 0.008n^2 - 0.008n + 0.196\\
w_{n+1} &= \tfrac{1.1}{0.9}w_n - \tfrac{0.008}{0.9}n^2 - \tfrac{0.008}{0.9}n + \tfrac{0.196}{0.9}
\end{aligned}$$

(c) We use the formula from 3b

$$\begin{aligned}
w_0 &= 2\\
w_1 &= \tfrac{1.1}{0.9} \cdot w_0 - \tfrac{0.008}{0.9} \cdot 0^2 - \tfrac{0.008}{0.9} \cdot 0 + \tfrac{0.196}{0.9}\\
&= \tfrac{1.1}{0.9} \cdot 2 + \tfrac{0.196}{0.9}\\
&= 2.662222\\
w_2 &= 3.453827\\
w_3 &= 4.385789
\end{aligned}$$

The absolute error is equal to: $|y(0.6) - w_3| = |4.3821188 - 4.385789| = 0.00366995$

(d) When we compare the errors of the three previous methods, we see the Trapezoidal rule method has the smallest error.

4. (a) $\bar{w}_{n+1} = w_n + hf(t_n, w_n)$

$w_{n+1} = w_n + \frac{h}{2}[f(t_n, w_n) + f(t_{n+1}, \bar{w}_{n+1})]$

(b) $h = \frac{2-0}{10} = 0.2$, $t_n = 0.2n$, $w_0 = 2$

$$\begin{aligned}
\bar{w}_{n+1} &= w_n + hf(t_n, w_n)\\
&= 1.2w_n - 0.008n^2 + 0.2 \qquad \text{see question about Euler Forwards}\\
w_{n+1} &= w_n + \tfrac{h}{2}[w_n - t_n^2 + 1 + \bar{w}_{n+1} - t_{n+1}^2 + 1]\\
&= w_n + 0.1(w_n - 0.04n^2 + 1 + 1.2w_n - 0.008n^2 + 0.2 - 0.04(n+1)^2 + 1)\\
&= w_n + 0.1w_n - 0.004n^2 + 0.1 + 0.12w_n - 0.0008n^2 + 0.02 - 0.004n^2\\
&\quad -0.008n - 0.004 + 0.1\\
&= 1.22w_n - 0.0088n^2 - 0.008n + 0.216
\end{aligned}$$

(c) We use the formula from 4b

$$\begin{aligned}
w_0 &= 2\\
w_1 &= 1.22 \cdot 2 + 0.216\\
&= 2.656\\
w_2 &= 3.43952\\
w_3 &= 4.3610144
\end{aligned}$$

The absolute error is equal to: $|y(0.6) - w_3| = |4.3821188 - 4.3610144| = 0.021104$

Section 6.4

1. (a) $y' = \lambda y, \qquad y(0) = y_0$
   $$\begin{aligned} w_{n+1} &= w_n + hf(t_n, w_n) \\ &= w_n + h(\lambda w_n) \\ &= (1 + h\lambda)w_n \end{aligned}$$
   (b) $1 + h\lambda$

2. (a) $y' = \lambda y, \qquad y(0) = y_0$
   $$\begin{aligned} w_{n+1} &= w_n + hf(t_{n+1}, w_{n+1}) \\ &= w_n + h\lambda w_{n+1} \\ w_{n+1} - h\lambda w_{n+1} &= w_n \\ (1 - h\lambda)w_{n+1} &= w_n \\ w_{n+1} &= \tfrac{1}{1-h\lambda} w_n \end{aligned}$$
   (b) $\frac{1}{1-h\lambda}$

3. (a) $y' = \lambda y, \qquad y(0) = y_0$
   $$\begin{aligned} w_{n+1} &= w_n + \tfrac{h}{2}[f(t_n, w_n) + f(t_{n+1}, w_{n+1}) \\ &= w_n + \tfrac{h}{2}[\lambda w_n + \lambda w_{n+1}] \\ &= (1 + \tfrac{h}{2}\lambda)w_n + \tfrac{h\lambda}{2} w_{n+1} \\ (1 - \tfrac{h\lambda}{2})w_{n+1} &= (1 + \tfrac{h\lambda}{2})w_n \\ w_{n+1} &= \tfrac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}} w_n \end{aligned}$$
   (b) $\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}$

4. (a) $y' = \lambda y, \qquad y(0) = y_0$
   $$\begin{aligned} \bar{w}_{n+1} &= w_n + hf(t_n, w_n) \\ &= w_n + h\lambda w_n \\ &= (1 + h\lambda)w_n \\ w_{n+1} &= w_n + \tfrac{h}{2}[f(t_n, w_n) + f(t_{n+1}, barw_{n+1}) \\ &= w_n + \tfrac{h}{2}[\lambda w_n + \lambda(1 + h\lambda)w_n] \\ &= w_n + \tfrac{h}{2}\lambda w_n + \tfrac{h}{2}(\lambda + h\lambda^2)w_n \\ &= (1 + \tfrac{h}{2}\lambda + \tfrac{h}{2}\lambda + \tfrac{h^2}{2}\lambda^2)w_n \\ &= (1 + h\lambda + \tfrac{1}{2}h^2\lambda^2)w_n \end{aligned}$$
   (b) $1 + h\lambda + \frac{1}{2}h^2\lambda^2$

5. (a) We have $f(t, y) = -5y^2 + 4$. Linearization gives:
   $$y' = f(\hat{t}, \hat{y}) + (y - \hat{y})\frac{\partial f}{\partial y}(\hat{t}, \hat{y}) + (t - \hat{t})\frac{\partial f}{\partial t}(\hat{t}, \hat{y})$$
   So we have $\lambda = \frac{\partial f}{\partial y}(\hat{t}, \hat{y}) = -10\hat{y}$

   (b) The initial value problem is stable when $\lambda \leq 0$.
   $\lambda = -10\hat{y} \leq 0$ for $\hat{y} \geq 0$ and that was given, so the problem is stable.

   (c) Euler Forwards is stable when $|Q(h\lambda)| = |1 + h\lambda| \leq 1$.
   $$\text{So} \quad \begin{array}{ccccc} & & |1 - 10\hat{y}h| & \leq & 1 \\ -1 & \leq & 1 - 10\hat{y}h & \leq & 1 \\ -2 & \leq & -10\hat{y}h & \leq & 0 \\ \frac{1}{5\hat{y}} & \geq & h & \geq & 0 \end{array}$$
   So stable for $h \leq \frac{1}{5\hat{y}}$

6. (a) $f(t, y) = -20y^2 + 4$
   $\lambda = \frac{\partial f}{\partial y}(\hat{t}, \hat{y}) = -40\hat{y}$

   (b) We have $\hat{y} \geq 0$, so $\lambda \leq 0$. So the initial value problem is stable.
   (c) Euler Backward is stable when $|Q(h\lambda)| = |\frac{1}{1-h\lambda}| \leq 1$.
   Because $40\hat{y}h \geq 0$, we have $1 + 40\hat{y}h \geq 1$ and so we have $0 < \frac{1}{1+40\hat{y}h} \leq 1$. And Euler Backwards is stable for all h.

7. The exact solution is given by $y_{j+1} = e^{h\lambda}y_j$.

8. We can write $e^{h\lambda}$ as it Taylor-expansion:
   $e^{h\lambda} = 1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + O(h^4)$.
   Now we get $\tau_{j+1} = \frac{e^{h\lambda}-Q(h\lambda)}{h} = \frac{1+h\lambda+O(h^2)-(1+h\lambda)}{h} = O(h)$.

9. $\tau_{j+1} = \frac{e^{h\lambda}-Q(h\lambda)}{h} = \frac{1+h\lambda+\frac{1}{2}h^2\lambda^2+O(h^3)-(\frac{1}{1-h\lambda})}{h} =$
   $\frac{1+h\lambda+\frac{1}{2}h^2\lambda^2+O(h^3)-(1+h\lambda+h^2\lambda^2+O(h^3))}{h} = \frac{-\frac{1}{2}h^2\lambda^2+O(h^3)}{h} = \frac{O(h^2)}{h} = O(h)$.

Section 6.5

1. (a) $w_{n+1} = w_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$
   with $k_1 = hf(t_n, w_n)$
   $k_2 = hf(t_n + \frac{1}{2}h, w_n + \frac{1}{2}k_1)$
   $k_3 = hf(t_n + \frac{1}{2}h, w_n + \frac{1}{2}k_2)$
   $k_4 = hf(t_n + h, w_n + k_3)$

   (b) $h = 0.2, t_n = 0.2n$

$$\begin{aligned}
w_0 &= 2\\
k_1 &= 0.2(w_0 - t_0^2 + 1)\\
&= 0.2(2 - 0.04 \cdot 0^2 + 1)\\
&= 0.6\\
k_2 &= 0.2(w_0 + 0.5 \cdot 0.6 - (t_0 + 0.5 \cdot 0.2)^2 + 1)\\
&= 0.2(2 + 0.3 - 0.01 + 1)\\
&= 0.658\\
k_3 &= 0.2(w_0 + 0.5 \cdot 0.658 - (t_0 + 0.5 \cdot 0.2)^2 + 1)\\
&= 0.2(2 + 0.329 - 0.001 + 1)\\
&= 0.6638\\
k_4 &= 0.2(w_0 + 0.6638 - (t_0 + 0.2)^2 + 1)\\
&= 0.2(2 + 0.6638 - 0.04 + 1)\\
&= 0.72476\\
w_1 &= 2 + \tfrac{1}{6}[0.6 + 2 \cdot 0.658 + 2 \cdot 0.6638 + 0.72476\\
&= 2.661393
\end{aligned}$$

$$\begin{aligned}
k_1 &= 0.2(2.661393 - 0.04 + 1)\\
&= 0.7243\\
k_2 &= 0.2(2.661393 + 0.5 \cdot 0.7243 - (0.2 + 0.1)^2 + 1)\\
&= 0.7867\\
k_3 &= 0.2(2.661393 + 0.5 \cdot 0.7867 - (0.2 + 0.1)^2 + 1)\\
&= 0.7929\\
k_4 &= 0.2(2.661393 + 0.7929 - (0.2 + 0.2)^2 + 1)\\
&= 0.8589\\
w_2 &= 2.661393 + \tfrac{1}{6}[0.7243 + 2 \cdot 0.7867 + 2 \cdot 0.7929 + 0.8589]\\
&= 3.451793
\end{aligned}$$

$$\begin{aligned}
k_1 &= 0.2(3.451793 - 0.16 + 1)\\
&= 0.85836\\
k_2 &= 0.2(3.451793 + 0.5 \cdot 0.85836 - (0.4 + 0.1)^2 + 1)\\
&= 0.92619\\
k_3 &= 0.2(3.451793 + 0.5 \cdot 0.92619 - (0.4 + 0.1)^2 + 1)\\
&= 0.93298\\
k_4 &= 0.2(3.451793 + 0.93298 - (0.4 + 0.2)^2 + 1)\\
&= 1.00495\\
w_3 &= 3.451793 + \tfrac{1}{6}[0.85836 + 2 \cdot 0.92619 + 2 \cdot 0.93298 + 1.00495]\\
&= 4.38207
\end{aligned}$$

3 steps $\Rightarrow t = 0.2 \cdot 3 = 0.6, y(0.6) = (0.6 + 1)^2 + e^{0.6} = 4.3821188$
$|y(0.6) - w_3| = |4.3821188 - 4.38207| = 5.0467 \cdot 10^{-5}$

2. (a) $f(t, y) = -4y^2 + 5$
   $\lambda = \frac{\partial f}{\partial y}(\hat{t}, \hat{y}) = -8\hat{y}$

   (b) For every $\hat{y} > 0$ we have $\lambda = -8\hat{y} < 0$, so the problem is stable.

   (c) RK4 is stable if $|Q(h\lambda)| = |1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4| \leq 1$
   $-1 \leq 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4 \leq 1$
   According the book, the left-hand side is true for every $h\lambda$, so also for every $-8h\hat{y}$. The righthand-side is true when $h < \frac{2.8}{|\lambda|} = \frac{2.8}{-8\hat{y}} = \frac{2.8}{8\hat{y}}$

Section 6.6

1. (a) $h = 0.1, t_n = 0.1n, w_0 = 2$

$$\begin{aligned}
w_1 &= w_0 + hf(t_0, w_0) &&= 2 + 0.1(-4w_0^2 + 5)\\
&= 2 + 0.1(-4 \cdot 4 + 5) &&= 0.9
\end{aligned}$$

(b) $h = 0.05, w_0 = 2$

$$\begin{aligned} w_1 &= w_0 + hf(t_0, w_0) && = 2 + 0.05(-4w_0^2 + 5) \\ &= 2 + 0.05(-4 \cdot 4 + 5) && = 1.45 \\ w_2 &= w_1 + hf(t_1, w_1) && = 1.45 + 0.05(-4w_1^2 + 5) \\ &= 1.45 + 0.05(-4 \cdot 1.45^2 + 5) && = 1.2795 \end{aligned}$$

(c) $\mathcal{O}(h)$, so $p = 1$.

(d) According to paragraph ”Global error and error proximation”we have e$(t, \frac{h}{2}) = y(t) - y(t, \frac{h}{2}) \approx \frac{1}{2^p-1}[y(t, \frac{h}{2}) - y(t, h)] = \frac{1}{2-1}[1.2795 - 0.9] = 0.3795$

2. (a) $h = 0.1, t_n = 0.1n, w_0 = 2$

$$\begin{aligned} \bar{w}_1 &= w_0 + hf(t_0, w_0) \\ &= 2 + 0.1(w_0 - t_0^2 + 1) \\ &= 2 + 0.1(2 - 0 + 1) \\ &= 2.3 \\ w_1 &= w_0 + \tfrac{h}{2}[f(t_0, w_0) + f(t_1, \bar{w}_1)] \\ &= 2 + 0.05[w_0 - t_0^2 + 1 + \bar{w}_1 - t_1^2 + 1] \\ &= 2 + 0.05[2 - 0 + 1 + 2.3 - (0.1)^2 + 1] \\ &= 2.3145 \end{aligned}$$

(b) $h = 0.05, t_n = 0.05n, w_0 = 2$

$$\begin{aligned} \bar{w}_1 &= w_0 + hf(t_0, w_0) \\ &= 2 + 0.05(w_0 - t_0^2 + 1) \\ &= 2 + 0.05(2 - 0 + 1) \\ &= 2.15 \\ w_1 &= w_0 + \tfrac{h}{2}[f(t_0, w_0) + f(t_1, \bar{w}_1)] \\ &= 2 + 0.025[w_0 - t_0^2 + 1 + \bar{w}_1 - t_1^2 + 1] \\ &= 2 + 0.025[2 - 0 + 1 + 2.15 - (0.05 \cdot 1)^2 + 1] \\ &= 2.1538125 \\ \bar{w}_2 &= w_1 + hf(t_1, w_1) \\ &= 2.1538125 + 0.05(2.1538125 - 0.05^2 + 1) \\ &= 2.311378 \\ w_2 &= w_1 + \tfrac{h}{2}[f(t_1, w_1) + f(t_2, \bar{w}_2)] \\ &= 2.1538125 + 0.025[2.1538125 - 0.05^2 + 1 + 2.311378 - 0.1^2 + 1] \\ &= 2.315130 \end{aligned}$$

(c) Order 2

(d) $p = 2$

e$(t, h) = y(t) - y(t, \frac{h}{2}) \approx \frac{1}{2^p-1}[y(t, \frac{h}{2}) - y(t, h)] = \frac{1}{2^2-1}[2.315317 - 2.3145] = 0.000272$

Section 6.7

1. (a) $\bar{w}_{n+1} = \bar{w}_n + h\bar{f}(t_n, \bar{w}_n)$

(b) $$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} u_1^n \\ u_2^n \end{bmatrix} + h \begin{bmatrix} -4u_1^n - 2u_2^n + \cos(0.1n) + 4\sin(0.1n) \\ 3u_1^n + u_2^n - 3\sin(0.1n) \end{bmatrix}$$

$$\begin{bmatrix} u_1^0 \\ u_2^0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0.1 \begin{bmatrix} -4 \cdot 0 - 2 \cdot -1 + \cos(0) + 4\sin(0) \\ 3 \cdot 0 + (-1) - 3\sin(0) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0.1 \begin{bmatrix} 2 + 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0.1 \begin{bmatrix} 0.3 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ -1.1 \end{bmatrix} \end{aligned}$$

1 step, so $t = 0.1 \cdot 1 = 0.1$
$u_1(0.1) = 2e^{-0.1} - 2e^{-0.2} + \sin(0.1) = 0.27205$
$u_2(0.1) = -3e^{-0.1} + 2e^{-0.2} = -1.07705$

(c) $|u_1(t) - u_1^1| = |0.27205 - 0.3| = 0.02795$
$|u_2(t) - u_2^1| = |-1.07705 + 1.1| = 0.02295$

2. (a) $x_1 = y$ with $x_1(0) = 1$
$x_2 = y'$ with $x_2(0) = 2$
So we get: $\begin{array}{rcl} x_1' & = & x_2 \\ x_2' & = & 2x_2 - x_1 + te^t - t \end{array}$

(b) $\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \end{bmatrix} = \begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix} + h \begin{bmatrix} x_2^n \\ 2x_2^n - x_1^n + 0.1ne^{0.1n} - 0.1n \end{bmatrix}$$

$$\begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 2 \\ 2 \cdot 2 - 1 + 0.1 \cdot 0 \cdot e^{0.1 \cdot 0} - 0.1 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 2.3 \end{bmatrix}$$

$\Rightarrow y \approx x_1 = 1.2$

(c) One step $\Rightarrow t = 0.1 \cdot 1 = 0.1$
$y(0.1) = 3e^{0.1} - 2 - 0.1 + \frac{1}{6} \cdot 0.1^3 \cdot e^{0.1} = 1.21570$
$|y(0.1) - y| = |1.21570 - 1.2| = 0.0157$

3. (a) $\mathbf{J}_n = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix}$

(b) $\mathbf{J}_n = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

(c) In vectorial form, we can write this system as:
$\mathbf{x}^{n+1} = \begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \end{bmatrix} = \begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix} + h \begin{bmatrix} x_2^n \\ -\sin x_1^n \end{bmatrix}$
Then $\mathbf{J}_n = \begin{pmatrix} 0 & 1 \\ -\cos x_1^n & 0 \end{pmatrix}$