Non-linear multigrid methods for Burgers' equation

The one-dimensional viscous Burgers' equation – named after the Dutch physicist Johannes Martinus Burgers (1895–1981) – is a common prototype of a nonlinear partial differential equation (PDE) from fluid mechanics. It describes the temporal evolution of a conserved quantity u subject to nonlinear convective transport as well as viscous effects. The conservative form of the equation reads

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \nu \frac{\partial^2 u}{\partial x^2}, \qquad t \in (0, T], \quad x \in (a, b)$$

which is complemented by compatible initial and boundary conditions conditions. Despite its simplicity, this equation features some of the challenges occurring in complex industrial flow models: the non-linearity of the convective term, the creation of discontinuous solutions (shock waves) as the viscosity coefficient $\nu \to +0$, etc. A good overview of the theory of scalar conservation laws is given in [LeV92]. The time-dependent one-dimensional Burgers' equation can be cast into an equivalent space-time stationary problem to be solved in the two-dimensional domain $(0, T] \times (a, b)$ with a finite difference scheme. This approach yields a nonlinear system of equations $A(u)u = b$ which needs to be solved by a suitable nonlinear solution algorithm. The main goal of this thesis is to compare the following two approaches:

1. Apply Newton's method to the nonlinear problem and solve the resulting linear problems

   $$u^{(m+1)} = u^{(m)} - J_A^{-1}(u^{(m)})A(u^{(m)}), \qquad m = 0, 1, \dots$$

   until convergence, where J_A denotes the Jacobi matrix of the nonlinear operator $A(u)$.

2. Apply a nonlinear multigrid (FAS) method to the nonlinear problem directly [vE02, Bri00].

Both approaches should be implemented in a MATLAB code and compared with respect to their nonlinear convergence behavior. A numerical study should be performed to answer the following questions:

- How does the nonlinear convergence behavior depend on the strength of the nonlinearity (magnitude of the viscosity parameter)?
- How does the smoothness/discontinuity of the solution profile influence the convergence behavior?
- Which method performs better if only an approximation to the Jacobian matrix is available?

Project details

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Prerequisites: Basic knowledge of numerical methods for partial differential equations and/or iterative solution techniques as well as programming skills (MATLAB/C/C++) is recommended.

References

[Bri00] *A Multigrid Tutorial*. SIAM, second edition, 2000.

[LeV92] Randall J. LeVeque. *Numerical Methods for Conservation Laws*. Birkhäuser Verlag, Basel, Boston, Berlin, second edition, 1992.

[vE02] Henson van Emden. Multigrid methods for nonlinear problems: An overview. Technical report, Center for Appleid Scientific Comuting, Lawrence Livermore National Laboratory, 2002.